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SIZE RAMSEY NUMBERS INVOLVING STARS

R.J. FAUDREE

Department of Mathematical Sciences, Memphis State University, Memphis, TN 38152, USA

J. SHEEHAN

Department of Mathematics, University of Aberdeen, Aberdeen, Scotland, UK

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We calculate some size Ramsey numbers involving stars. For example we prove that for $t \geq k \geq 2$ and n sufficiently large the size Ramsey number

$$r_q(K_{1,k}, K_t + \bar{K}_n) = \binom{k(t-1)+1}{2} + (k(t-1)+1)(n+k-1).$$

All graphs in this paper are finite, simple and undirected. Let F, G and H be graphs. The number of vertices and edges of a graph F will be denoted by $p(F)$ and $q(F)$ respectively. A graph $F \to (G, H)$ if every 2-colouring (say red and blue) of the edges of F produces either a 'red' G or a 'blue' H. Since all colourings of F below will be 2-colourings of the edges of F we shall simply refer to 'colourings' of F. *The Ramsey number*

$$r(G, H) = r_p(G, H) = \min\{p(F) : F \to (G, H)\},$$

the size Ramsey number

$$r_q(G, H) = \min\{q(F) : F \to (G, H)\},$$

and *the restricted size Ramsey number*

$$r_q^*(G, H) = \min\{q(F) : F \to (G, H),\ p(F) = r(G, H)\}.$$

Also if H is a subgraph of G, then G-H will denote the graph obtained from G by deleting the edges of H. Thus both G and G-H will have the same vertex set. $\bar{F}$ denotes the complement of F. Suppose G and H are distinct (disjoint) graphs. Then $G \cup H$ is their disjoint union and $G + H$ denotes the complement of $\bar{G} \cup \bar{H}$.

The term 'size Ramsey number' was introduced in [5] and further studied in [4], [6], [8] and [9].

A survey of the few results concerning this function can be found in [2]. The main results in this paper are for size Ramsey numbers involving stars. They include:

Theorem 1. *Let* $\alpha = k(n-1)+1$ $(k, n \geq 2)$. *Then*

$$r_q^*(K_{1,k}, K_n) = \begin{cases} \binom{\alpha}{2} - \binom{k}{2} & k \geq n \text{ or } k \text{ odd,} \\ \binom{\alpha}{2} - \frac{1}{2}k(n-1) & \text{otherwise.} \end{cases}$$

Theorem 2. $r_q(K_{1,2}, K_n) = r_q^*(K_{1,2}, K_n) = 2(n-1)^2$ $(n \geq 2)$.

Theorem 3. *For* $t \geq k \geq 2$ *and n sufficiently large*

$$r_q(K_{1,k}, K_t + \bar{K}_n) = r_q^*(K_{1,k}, K_t + \bar{K}_n) = \binom{\alpha}{2} + \alpha(n+k-1)$$

where $\alpha = k(t-1)+1$.

1. Preliminary notation

Let G be a graph and suppose $X, Y \subseteq V(G)$. Then $\langle X \rangle$ is the induced subgraph of G with vertex set X. $E(X, Y)$ is the set of edges with one end vertex in X and the other in Y. Let $x \in X$. Then $N(x)$ is the set of *neighbours* of x and $N(X) = \bigcap_{x \in X} N(x)$. $N_Y(x) = N(x) \cap Y$. The degree $d(x)$ of x is by definition $|N(x)|$ and $d_Y(x) = |N_Y(x)|$. $\delta(G)$, $\Delta(G)$ denote respectively the *minimum* and *maximum* degrees of G. Most of this terminology is by now fairly standard. In general terminology not defined here follows either [1] or [7].

2. Theorems 1 and 2

Lemma 1. *Let* $k \geq 2$. *Let G be a graph with* $q(G) \geq \binom{k}{2}+1$. *Then either*
(i) *G contains an induced subgraph with* $k+1$ *vertices and minimal degree* ≥ 1 *or*
(ii) *G contains a matching M with* $|M| = \binom{k}{2}+1$.

Proof. We use induction on k. The lemma is trivially true for $k=2$. Suppose now that G is a graph with

$$q(G) \geq \binom{k}{2} + 1 \quad (k \geq 3)$$

and the proposition holds for all $2 \leq s < k$. Let $v \in V(G)$. Then $d(v) \leq k-1$ otherwise Lemma 1(i) holds. Furthermore there exists $v \in V(G)$ such that $d(v) \geq 2$ otherwise Lemma 1(ii) holds. Therefore choose v so that $2 \leq d(v) \leq k-1$. Write $G^* = G - v$. Then $q(G^*) \geq \binom{k-1}{2}+1$ and so, by induction, there exists

$Y \subseteq V(G^*)$ such that, either

(a) $|Y| = k$, $\delta(\langle Y \rangle) \geq 1$, or

(b) $\langle Y \rangle \cong sK_2$, $s = \binom{k-1}{2} + 1$.

Firstly assume $|N(v) \cap Y| \neq 0$. If Y is of type (a), then write $X = Y \cup \{v\}$. Suppose now Y is of type (b). If k is odd (and hence $k \geq 3$) let X be the set of $k+1$ vertices incident to some set of $\frac{1}{2}(k+1)$ disjoint edges contained in $\langle Y \rangle$. If k is even let $u \in N(v) \cap Y$ and choose a set M of $\frac{1}{2}k$ disjoint edges contained in $\langle Y \rangle$ including an edge incident to u. Let X be the set of $k+1$ vertices consisting of those vertices incident to M together with v. In all cases $\langle X \rangle$ satisfies Lemma 1(i).

Now assume $|N(v) \cap Y| = 0$. Let $v_1, v_2 \in N(v)$. Suppose Y is of type (a). Choose any vertex $u \in Y$ and write $X^* = (Y - u) \cup \{v, v_1\}$. We may suppose X^* does not satisfy Lemma 1(i). Hence there exists $x \in X^*$ such that $|N(x) \cap X^*| = 0$. Since Y is of type (a) it follows that $x \in Y - u$ and $xu \in E(G)$. So we have proved that any vertex $u \in Y$ is incident to some vertex $x \in Y$ with $d_Y(x) = 1$. In particular this is true for the vertex x and so $d_Y(u) = 1$. Therefore k is even and $\langle Y \rangle \cong (\frac{1}{2}k)K_2$. Let Y^* be the set of vertices incident to some subset of $\frac{1}{2}(k-2)$ edges contained in $\langle Y \rangle$. Let $X = Y^* \cup \{v, v_1, v_2\}$. Then $\langle X \rangle$ satisfies Lemma 1(i). This completes the induction. □

Lemma 2. *Let $k \geq 3$. Let G be a graph with $q(G) \geq \binom{k}{2} + 1$. If k is odd, then G contains an induced subgraph with $k+1$ vertices and minimal degree ≥ 1.*

Proof. We may suppose, by Lemma 1, that G contains a matching M with $|M| = \binom{k}{2} + 1$. Let X be the set of vertices incident to a subset of $\frac{1}{2}(k+1)$ elements of M. Then $\langle X \rangle$ is an appropriate induced subgraph. □

Lemma 3. *Let G be a graph with $p(G) = k(n-1) + 1$ and*

$$q(G) \leq \binom{k(n-1)+1}{2} - \binom{k}{2} - 1 \quad (k, n \geq 2).$$

Then, if either (i) *k is odd or* (ii) *$k \geq n$, $G \nrightarrow (K_{1,k}, K_n)$.*

Proof. If $k \geq n$, then

$$2\left(\binom{k}{2} + 1\right) > (k(n-1) + 1) = p(\bar{G})$$

and so $\bar{G}$ cannot contain a matching M with $|M| = \binom{k}{2} + 1$. Hence by Lemmas 1 and 2 if k is odd or $k \geq n$, $\bar{G}$ contains an induced subgraph $\langle X \rangle$ with $|X| = k+1$ and minimal degree ≥ 1. We now colour G. Partition $V(G)$ into subsets $X_0, X_1, \ldots, X_{n-2}$ so that $X_0 = X$ and $|X_i| = k$ $(i = 1, 2, \ldots, n-2)$. Colour every edge of $\langle X_i \rangle$ red $(i = 0, 1, 2, \ldots, n-2)$ and all other edges of G blue. Then there is no red $K_{1,k}$ and no blue K_n. Hence $G \nrightarrow (K_{1,k}, K_n)$. □

Lemma 4. *Let G be a graph with $p(G)=k(n-1)+1$ and*

$$q(G)\leqslant\binom{k(n-1)+1}{2}-\tfrac{1}{2}k(n-1)-1 \quad (k, n\geqslant 2).$$

Then if $k<n$, $G\not\to(K_{1,k}, K_n)$.

Proof. Assume $\bar{G}$ does not contain an induced subgraph with $k+1$ vertices and minimal degree $\geqslant 1$. Since $q(\bar{G})\geqslant\frac{1}{2}(k(n-1))+1\geqslant\binom{k}{2}+1$, by Lemmas 1 and 2, k is even and there exists $Y\subseteq V(\bar{G})$ with $\langle Y\rangle\cong sK_2$, $s\geqslant\binom{k}{2}+1$. Choose Y so that s is as large as possible. If $v\in V(G)\setminus Y$, then $|N(v)\cap Y|=0$ otherwise (see the proof of Lemma 1) $\bar{G}$ contains the aforementioned induced subgraph. On the other hand $|N(v)\cap(V(G)\setminus Y)|=0$ otherwise the maximality of s is contradicted. Hence $d(v)=0$. Therefore $s=q(\bar{G})\geqslant\frac{1}{2}k(n-1)+1$ and $q(\bar{G})\geqslant k(n-1)+2$ which is a contradiction. Hence there exists $X\subseteq V(\bar{G})$ such that $|X|=k+1$ and $\delta(\langle X\rangle)\geqslant 1$, where $\langle X\rangle$ is regarded here as an induced subgraph of $\bar{G}$. We now colour G exactly as in Lemma 3 and likewise prove that $G\not\to(K_{1,k}, K_n)$. □

Theorem 1. *Let $\alpha=k(n-1)+1$ $(k, n\geqslant 2)$. Then*

$$r_q^*(K_{1,k}, K_n)=\begin{cases}\binom{\alpha}{2}-\binom{k}{2} & k\geqslant n \text{ or } k \text{ odd},\\[2mm] \binom{\alpha}{2}-\frac{1}{2}k(n-1) & \text{otherwise}.\end{cases}$$

Proof. Let $F\cong K_\alpha$ and let H be a subgraph of F with $H\cong K_k$. Let $G=F-H$. We prove that $G\to(K_{1,k}, K_n)$. The proof is by induction on n. The induction hypothesis is easily verified when $n=2$. So assume $n\geqslant 3$. Suppose G is coloured and there is no red $K_{1,k}$. Let $v\in V(G)\setminus V(H)$. Let $R(v)$ be the red neighbourhood of v. Then $G-(R(v)\cup\{v\})$ contains a subgraph (not necessarily induced) isomorphic to $K_{k(n-2)+1}-K_k$. By the induction hypothesis since in the induced colouring of this subgraph there is no red $K_{1,k}$ it follows that there is a blue K_{n-1}. This blue K_{n-1} is in the blue neighbourhood of v and so in the original colouring there is a blue K_n. Hence $G\to(K_{1,k}, K_n)$. Therefore, by Lemma 3,

$$r_q^*(K_{1,k}, K_n)=\binom{\alpha}{2}-\binom{k}{2} \quad (k\geqslant n \text{ or } k \text{ odd}).$$

So now suppose that $k\leqslant n$ and k is even. Let G be the graph obtained from K_α by deleting a maximal matching i.e. $\bar{G}\cong K_1\cup sK_2$ where $s=\frac{1}{2}(\alpha-1)$. We prove, again by induction on n, that $G\to(K_{1,k}, K_n)$. Again the induction hypothesis is easily verified when $n=2$. So suppose $n\geqslant 3$. Let v be the vertex of maximum degree in G (i.e. v is not incident to any edge in the deleted matching). Suppose G is coloured and there is no red $K_{1,k}$. Then $G-(R(v)\cup\{v\})$ contains a subgraph isomorphic to H where $\bar{H}\cong K_1\cup s_0K_2$ and $s_0=\frac{1}{2}(n-2)k$. By induction in the induced colouring of H there is a blue K_{n-1} and hence a blue K_n in the original colouring of G. Hence $G \to (K_{1,k}, K_n)$. So, by Lemma 4,

$$r_q^*(K_{1,k}, K_n) = \binom{\alpha}{2} - \tfrac{1}{2}k(n-1). \qquad \square$$

It is natural to conjecture (although obviously foolhardy):

Conjecture 1. $r_q(K_{1,k}, K_n) = r_q^*(K_{1,k}, K_n)$.

We can prove:

Theorem 2. $r_q(K_{1,2}, K_n) = r_q^*(K_{1,2}, K_n) = 2(n-1)^2 \quad (n \geq 2)$.

Proof. Let M be any maximal matching in K_{2n-1}. Let $G = K_{2n-1} - M$ be the graph obtained from K_{2n-1} by deleting M (i.e. $\bar{G} \cong K_1 \cup sK_2$, $s = n-1$). This is a very slight abuse of our earlier notation. We show $G \to (K_{1,2}, K_n)$. Colour the edges of G so that there is no red $K_{1,2}$. Then the graph H with $V(H) = V(G)$ whose edge set consists of M together with the edges coloured red is a disjoint union of even cycles and paths with at least one path having an odd number (possibly 1) of vertices. Hence $\bar{H}$ contains a subgraph isomorphic to K_n. But $E(\bar{H})$ is precisely the set of blue edges in the colouring of G and so there is a blue K_n. We have proved $G \to (K_{1,2}, K_n)$ and $r_q(K_{1,2}, K_n) \leq 2(n-1)^2$.

Supppose G is a graph such that $G \to (K_{1,2}, K_n)$ and $p(G) = 2n-1$. We prove, by induction on n, that $\delta(G) \geq 2n-3$. We may obviously assume that G is edge-minimal. The induction step for $n = 2$ is easily verified. Hence suppose the result is true for all s, $3 \leq s < n$. Clearly we may suppose $\delta(G) \geq 2$ since if there existed a vertex of degree ≤ 1 then G must contain a proper subgraph G^* with $G^* \to (K_{1,2}, K_n)$ and $p(G^*) = 2n-1$ contradicting edge-minimality. Let $u \in V(G)$ and $v, w \in N(u)$. Let $G^* = G - \{v, w\}$. Colour G^* so that there is no red $K_{1,2}$. Extend this colouring to G by colouring the edge vw (if it exists) red and all so far uncoloured edges blue. Then there is a blue K_n containing at most one of v and w. Hence in the colouring of G^* there is a blue K_{n-1}. Therefore $G^* \to (K_{1,2}, K_{n-1})$ and $p(G^*) = 2(n-1)-1$. By induction, $\delta(G^*) \geq 2n-5$. Hence $d(u) \geq 2n-3$ and so, since u was chosen arbitrarily, $\delta(G) \geq 2n-3$. This completes the induction. As a consequence if $G \to (K_{1,2}, K_n)$ and $p(G) = 2n-1$, $q(G) \geq 2(n-1)^2$.

Now suppose G is a graph such that $G \to (K_{1,2}, K_n)$ and $p(G) \geq 2n$. We prove by induction on n that in this case $q(G) \geq 2(n-1)^2$. We may again assume that G is edge-minimal. The result is trivially true when $n = 2$ so assume it is true for all s, $3 \leq s < n$. If $\Delta(\bar{G}) \leq 1$, then $q(G) \geq 2(n-1)^2$. Hence we may suppose $\Delta(\bar{G}) \geq 2$ and in particular we may choose vertices u, v, w such that $uv, uw \in E(\bar{G})$. Firstly we prove that the induction is complete or $\bar{G}$ contains either a C_3 or a C_4 (C_n denotes a cycle of length n). Let $G^* = G - \{u, v, w\}$. Colour G^* so that there is no red $K_{1,2}$. Extend this to a colouring of G by colouring the edge vw (if it exists) red and all other so far uncoloured edges blue. Then since there is no red $K_{1,2}$ there is a blue K_n which contains at most one of u, v and w. Hence in the colouring of G^* there is a blue K_{n-1}. So $G^* \to (K_{1,2}, K_{n-1})$ and $p(G^*) \geq 2n-3$. If $p(G^*) = 2n-3$ then, by the previous paragraph, $p(G^*) \geq 2(n-2)^2$. If $p(G^*) \geq 2n-2$ then, by induction, $q(G^*) \geq 2(n-2)^2$. Now suppose $\bar{G}$ contains neither a C_3 or a C_4. Then no vertex of $\bar{G}^*$ is adjacent to two of u, v and w. Hence every vertex of G^* is adjacent to at least two of u, v and w. Hence

$$q(G) \geq q(G^*) + 2(2n-3) \geq 2(n-2)^2 + 2(2n-3) = 2(n-1)^2.$$

Hence we may suppose $\bar{G}$ contains either a C_3 or C_4. Secondly we prove that either the induction is complete or $\bar{G}$ contains a C_4. Suppose $\bar{G}$ contains no C_4. Then select u, v, w so that $uv, uw, vw \in E(\bar{G})$ and again define $G^* = G - \{u, v, w\}$. Then since $\bar{G}$ contains no C_4 no vertex of $\bar{G}^*$ is adjacent to two of u, v and w and so by exactly the same argument $q(G) \geq 2(n-1)^2$. Therefore we may assume $\bar{G}$ contains a C_4. Let (v_1, v_2, v_3, v_4) be a 4-cycle in $\bar{G}$. Let $G^* = G - \{v_1, v_2, v_3, v_4\}$. Colour G^* so that there is no red $K_{1,2}$. Now extend this colouring to G by colouring red the edges v_1v_3 and v_2v_4 (when either or both exist) Complete the colouring of G by colouring all other so far uncoloured edges blue. Since there is no red $K_{1,2}$ there is a blue K_n containing at most one of the v_i's. Hence in the colouring of G^* there is a blue K_{n-1} and $G^* \to (K_{1,2}, K_{n-1})$. Now $p(G^*) \geq 2n-4$. Since $r(K_{1,2}, K_{n-1}) = 2n-3$, $p(G^*) \geq 2n-3$ and if $p(G^*) = 2n-3$ we have proved above that $q(G^*) \geq 2(n-2)^2$. On the other hand if $|V(G^*)| \geq 2(n-1)$ the induction hypothesis yields $q(G^*) \geq 2(n-2)^2$. We may assume because of edge-minimality that each vertex of G must be in some K_n. Hence $\delta(G) \geq n-1$ and

$$q(G) \geq q(G^*) + \sum_{i=1}^{4} d(v_i) - 2 \geq 2(n-2)^2 + 4(n-1) - 2 = 2(n-1)^2.$$

This completes the induction argument. □

Theorem 1 for $k=3$, $n=4$ and Theorem 2 for $n=3,4$ are quoted in [6]. Conjecture 1 becomes, in the special case $n=3$, $k \geq 3$:

Conjecture 2.

$$r_q(K_{1,k}, K_3) = r_q^*(K_{1,k}, K_3) = \binom{2k+1}{2} - \binom{k}{2} \quad (k \geq 3).$$

Not without some considerable effort (the result is quoted in [6]) we have proved that $r_q(K_{1,3}, K_3) = 18$ verifying the conjecture in this case. As a first step in a proof of Conjecture 2 we would need to prove an analogue to Lemma 3. Namely that if G is a graph with $p(G) = 2k+2$, $\delta(G) > 0$ and

$$q(G) \leq \binom{2k+1}{2} - \binom{k}{2} - 1 \quad (k \geq 4),$$

then $G \not\to (K_{1,k}, K_3)$. We have made progress towards a proof of this.

3. Theorem 3

Theorem 3. *Suppose $t \geqslant k \geqslant 2$ and write $\alpha = k(t-1)+1$. Then, for n sufficiently large,*

$$r_q(K_{1,k}, K_t + \bar{K}_n) = r_q^*(K_{1,k}, K_t + \bar{K}_n) = \binom{\alpha}{2} + \alpha(n+k-1).$$

Proof. We give only the briefest of outlines. Firstly

$$K_\alpha + \bar{K}_{n+k-1} \rightarrow (K_{1,k}, K_t + \bar{K}_n).$$

This is easily seen since $r(K_{1,k}, K_t) = k(t-1)+1$ (see [3]). Incidentally this is the unique arrowing graph with this number of edges.

Secondly suppose that $G \rightarrow (K_{1,k}, K_t + \bar{K}_n)$. Let $\mathcal{H} = \{v \in V(G): d(v) \geqslant n+t-1\}$. These are the 'high' vertices. We now use an almost standard 'high-low argument' (see [5]) together with Lemmas 3 and 4 to complete the proof.

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